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# Properties of $\beta$-Splines

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## 1. INTRODUCTION

It has been found that $B$-splines play a useful role in computer-aided design. One of their advantages is their "variation-diminishing" property [8] which implies that the design curve cannot cross any given straight line more often than does the sequence of control vertices. (A more detailed explanation is given in Sect. 2.) In [1] Barsky generalised $B$-splines to $\beta$-splines, which preserve the geometric smoothness of the design curve while allowing the continuity conditions on the spline functions at the knots to be varied by certain parameters, thus giving greater flexibility.

The main purpose of this paper is to show that $\beta$-splines also have the variation-diminishing property. We also show that varying the parameters at one knot, keeping the others fixed, has only a local effect on the design curve. This may be of interest since in [2, p. 29] the authors reject this method of gaining greater flexibility through fear that the effects might not be local.

In Section 2 we review briefly the use of $B$-splines and $\beta$-splines in constructing design curves. The $\beta$-splines so used are cubic with simple knots and two parameters at each knot. However, the results of this paper hold equally well for splines of arbitrary degree $n$ with multiple knots and up to $2n-1$ parameters at each knot. In Section 3 we study these spaces of splines and show under precisely what conditions they are suitable for interpolation. In Section 4 we introduce the basis of generalised $\beta$-splines and prove their variation-diminishing and other properties.

In Section 6 we consider how these generalised $\beta$-splines can be used to construct a design curve. It is natural to investigate the geometric smoothness of this curve. To this end we derive, in Section 5, necessary and sufficient conditions for geometric continuity of arbitrary degree. Finally. in Section 6, we give an explicit formula for the cubic $\beta$-splines with second



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degree geometric continuity and uniformly spaced knots, generalising a formula given in [2] in which the parameters were restricted to be the same at each knot.

## 2. $B$-Splines and $\beta$-Splines

In this section we shall consider briefly the use in constructing design curves of cubic $B$-splines and their generalisation by Barsky to $\beta$-splines. For simplicity we consider curves in $\mathbb{R}^2$, but everything extends easily to $\mathbb{R}^3$. For further details and extensions the reader is referred to [3].

For $i=0,..., k-1$ et $V_i=(x_i, y_i)$ be points in $\mathbb{R}^2$ which we refer to as *control vertices*. The polygon formed by joining consecutive control vertices by straight line segments is called the *control polygon*. The problem we are concerned with is to construct a smooth curve, called the *design curve*, which in some sense approximates, and mimics in shape, the control polygon. Moving the control vertices will then provide a simple means of modifying the design curve into a visually desirable form. One way to do this is to use cubic $B$-splines, as we now describe.

Take $k>3$ and a strictly increasing sequence $\mathbf{t}=(t_j)_0^{k+3}$ whose elements we shall call *knots*. For $i=0,..., k-1$, there are unique functions $N_i$ in $C^2(\mathbb{R})$ satisfying the following properties.

$$N_i \mid [t_j, t_{j+1}] \text{ is a cubic polynomial} \qquad (j=0,..., k+2), \tag{2.1}$$

$$N_i(t)=0 \qquad \text{for } t \notin (t_i, t_{i+4}), \tag{2.2}$$

$$N_i(t)>0 \qquad \text{for } t \in (t_i, t_{i+4}), \tag{2.3}$$

$$\sum_{i=0}^{k-1} N_i(t)=1 \qquad (t_3 \leqslant t \leqslant t_k). \tag{2.4}$$

The functions $N_i$ are the well-known normalised cubic $B$-splines. We now construct a design curve $Q$ in $\mathbb{R}^2$, defined parametrically by

$$Q(t)=(Q_1(t), Q_2(t))=\sum_{i=0}^{k-1} V_i N_i(t) \qquad (t_3 \leqslant t \leqslant t_k). \tag{2.5}$$

As the $B$-splines are simple to compute and manipulate, so is the design curve. In addition the curve has the following desirable properties.

(a) *Smoothness.* As the $B$-splines are $C^2$, the design curve satisfies second degree geometric continuity, i.e., the curve, the unit tangent vector and the curvature vector are continuous functions of the parameter $t$. This ensures that the design curve is visually smooth.

(b) *Approximation.* From (2.2), (2.4), and (2.5) we see that for $t$ in $(t_j, t_{j+1})$, $3 \leqslant j \leqslant k-1$,

$$Q(t) = \sum_{i=j-3}^{j} V_i N_i(t), \qquad \sum_{i=j-3}^{j} N_i(t) = 1. \tag{2.6}$$

Thus $Q(t)$ lies in the convex hull of $V_{j-3}$, $V_{j-2}$, $V_{j-1}$, and $V_j$.

(c) *Shape preservation.* The design curve has the interesting property that it crosses any given straight line no more often than does the control polygon. (In particular, if the control polygon is convex, then the design curve is convex). To show this we need the well-known variation-diminishing property of $B$-splines [8] which we now describe.

For any real vector $v = (v_0, \ldots, v_r)$, we let $S^- v$ denote the number of strict sign changes in the sequences $v$, i.e., the number of sign changes when zero entries in $v$ are ignored. For any function $f$ on $\mathbb{R}$ we shall denote by $V(f)$ the number of times that $f$ changes sign; i.e., $V(f)$ is the supremium of $S^-(f(x_1), \ldots, f(x_r))$ over all increasing sequences $(x_1, \ldots, x_r)$ for all $r$. The $B$-splines $N_0, \ldots, N_{k-1}$ possess the remarkable property that for any numbers $a_0, \ldots, a_{k-1}$,

$$V(a_0 N_0 + \cdots + a_{k-1} N_{k-1}) \leqslant S^-(a_0, \ldots, a_{k-1}). \tag{2.7}$$

We now return to the design curve (2.5) and take any straight line $L$ in $\mathbb{R}^2$, say

$$L = \{(x, y) : ax + by + c = 0\}.$$

Recalling (2.4) we see that

$$\begin{aligned} aQ_1(t) + bQ_2(t) + c &= a \sum_{i=0}^{k-1} x_i N_i(t) + b \sum_{i=0}^{k-1} y_i N_i(t) + c \sum_{i=0}^{k-1} N_i(t) \\ &= \sum_{i=0}^{k-1} (ax_i + by_i + c)\, N_i(t). \end{aligned}$$

Then (2.7) tells us that

$$V(aQ_1 + bQ_2 + c) \leqslant S^-((ax_i + by_i + c)_0^{k-1});$$

i.e., $Q$ crosses $L$ no more often than does the control polygon.

(d) *Local control.* An important property of the design curve is that changing a control vertex changes the curve only locally. To be precise, we see from (2.2) that changing $V_i$ will affect $Q(t)$ only for $t$ in $(t_i, t_{i+4})$.

We now consider the generalisation of the above to $\beta$-splines. In this thesis [1], Barsky made the simple and important observation that for second degree geometric continuity of a curve $Q(t)=(Q_1(t), Q_2(t))$, it is not necessary that $Q_1$ and $Q_2$ are $C^2$. To be precise, suppose $Q$ is $C^2$ on $[a-\delta, a]$ and on $[a, a+\delta]$ for some $a$ and $\delta>0$. Then Barsky showed that a necessary and sufficient condition for $Q$ to have second degree geometric continuity on $[a-\delta, a+\delta]$ is that for some numbers $\beta_1>0$ and $\beta_2$,

$$\begin{aligned} Q(a^+) &= Q(a^-), \\ Q'(a^+) &= \beta_1 Q'(a^-), \\ Q''(a^+) &= \beta_1^2 Q''(a^-) + \beta_2 Q'(a^-). \end{aligned} \tag{2.8}$$

Barsky then constructed for numbers $\beta_1>0$ and $\beta_2 \geqslant 0$, and for a uniform sequence $t_j=j$ $(j=0,..., k+3)$, functions $N_i$ $(i=0,..., k-1)$ satisfying (2.1)–(2.4) and at any knot $t_j$ $(j=0,..., k+3)$,

$$\begin{aligned} N_i(t_j^+) &= N_i(t_j^-), \\ N_i'(t_j^+) &= \beta_1 N_i'(t_j^-), \\ N_i''(t_j^+) &= \beta_1^2 N_i''(t_j^-) + \beta_2 N_i'(t_j^-). \end{aligned} \tag{2.9}$$

He called these function *$\beta$-splines*. If $\beta_1=1$, $\beta_2=0$ they reduce to the normalised cubic $B$-splines. For these $\beta$-splines we can define a design curve by (2.5), which is seen to satisfy (a), (b), and (d) as before. In this paper we shall show that the $\beta$-splines also satisfy the variation-diminishing property (2.7) and thus the design curve is shape preserving as in (c).

The generalisation to $\beta$-splines is useful because it allows one to vary the shape of the design curve by varying the parameters $\beta_1$ and $\beta_2$ (called *bias* and *tension*, respectively). One can gain still further flexibility by allowing different values of $\beta_1$ and $\beta_2$ for different knots $t_j$. In [2, p. 29] the authors reject this method through fear that the effect of varying the parameters at only one knot would not be local. However, we shall show that varying the parameters at a single knot does have only a local effect on the design curve.

## 3. The Spline Functions

The results we shall prove for linear combinations of cubic $\beta$-splines hold for a much broader class of functions which we shall now introduce.

Take $n \geqslant 0$, $k \geqslant 0$, and a nondecreasing sequence $\mathbf{t}=(t_i)_0^{n+k}$ with multiplicities at most $n+1$, i.e., $t_0 \leqslant \cdots \leqslant t_{n+k}$ and $t_i < t_{i+n+1}$ for $i=0,...,$

$k-1$. We shall refer to the elements of $\mathbf{t}$ as *knots*, and we shall define a space of "spline functions" which are polynomials of degree $\leqslant n$ between knots and "tied together" in a certain manner at the knots.

With each point $t$ in $\{t_0, ..., t_{n+k}\}$ satisfying $t_0 < t < t_{n+k}$ we associate vectors $B(t)$ and $\Gamma(t)$ as follows. If $t$ has multiplicity $\mu$ in $\mathbf{t}$, then $B(t) = (B_j(t))_0^{n-\mu}$, $\Gamma(t) = (\Gamma_j(t))_1^{n-\mu}$, where $B_j(t) > 0$ $(j = 0, ..., n-\mu)$ and $\Gamma_j(t) \geqslant 0$ $(j = 1, ..., n-\mu)$. We denote by $\zeta(\mathbf{t}, B, \Gamma)$ (or $\zeta(t)$ or $\zeta$, if unambiguous) the space of functions $f$ satisfying the following properties:

(a) If $t_i < t_{i+1}$, then $f \mid [t_i, t_{i+1})$ is a polynomial of degree $\leqslant n$.

(b) $f$ vanishes outside $[t_0, t_{n+k})$.

(c) If $t$ has multiplicity $\mu$ in $\mathbf{t}$ and either $t = t_0$ or $t = t_{n+k}$, then $f$ is $C^{n-\mu}$ in a neighbourhood of $t$.

(d) If $t$ has multiplicity $\mu$ in $\mathbf{t}$ and $t_0 < t < t_{n+k}$, then

$$\begin{aligned} f^{(j)}(t^+) &= B_j(t) f^{(j)}(t^-) + \Gamma_j(t) f^{(j-1)}(t^-) \qquad (j = 1, ..., n-\mu), \\ f(t^+) &= B_0(t) f(t^-). \end{aligned} \tag{3.1}$$

(If $\mu = n+1$, then conditions (3.1) become vacuous.)

We shall refer to elements of $\zeta$ as spline functions (or splines) of degree $n$ with knots at $\mathbf{t}$ and parameters $B$, $\Gamma$. If $B_j(t) = 1$, $\Gamma_j(t) = 0$ for all possible $t$ and $j$, then these spline functions reduce to ordinary polynomial spline functions. If $n = 3$, $\mu = 1$, $B(t) = (1, \beta_1, \beta_1^2)$, $\Gamma(t) = (0, \beta_2)$ for each knot $t$, then (3.1) reduces to the conditions for second degree geometric continuity (see (2.9)). So in this case the splines reduce to linear combinations of the $\beta$-splines considered in Section 2.

We wish to derive a bound on the number of zeros of a function $f$ in $\zeta$, and so we first describe how to count these zeros. It will be useful to adopt the notation that $f(a)^+ = 1$, 0, or $-1$ if for small enough $\varepsilon > 0$, $f(x) > 0$, $= 0$ or $< 0$, respectively, on $(a, a+\varepsilon)$. Similarly we write $f(a)^- = 1$, 0, or $-1$ if for small enough $\varepsilon > 0$, $f(x) > 0$, $= 0$, or $< 0$ on $(a-\varepsilon, a)$.

Now suppose that for some point $a$, $f(a)^- \neq 0 \neq f(a)^+$ and $f(a^-) = \cdots = f^{(l-1)}(a^-) = 0 \neq f^{(l)}(a^-), f(a^+) = \cdots = f^{(r-1)}(a^+) = 0 \neq f^{(r)}(a^+)$. Then we say that $f$ has a *point zero* at $a$ of multiplicity $m$, where $m = \max(l, r)$ or $m = \max(l, r) + 1$, and $f(a)^- f(a)^+ = (-1)^m$.

Next suppose that for $a < b$, $f(a)^- \neq 0 \neq f(b)^+$ and $f(x) = 0$ $(a < x < b)$. Let $v$ denote the number of elements of $\mathbf{t}$ in $(a, b)$, i.e., $v = |\{i: a < t_i < b\}|$. Then we say that $f$ has an *interval zero* at $(a, b)$ of multiplicity $v + n + 2$.

Finally for any interval $I$ we let $Z(f \mid I)$ denote the total number of point zeros and interval zeros of $f$ in $I$, counted with multiplicity. We shall write $Z(f \mid (-\infty, \infty))$ simply at $Z(f)$.

THEOREM 1. *For any nontrivial function $f$ in $\zeta$ we have*

$$Z(f) \leqslant k-1.$$

Our proof of Theorem 1 will depend on an estimate for the number of zeros of polynomials called the Budan–Fourier theorem [6]. For a real vector $v=(v_0,..., v_r)$, we let $S^+v$ denote the maximal number of sign changes in the sequence $v$ achievable by appropriate assignment of signs to the zero entries of $v$. Then the Budan–Fourier theorem states that if $p$ is a polynomial of exact degree $r$ and $Z(p \mid (a, b))$ denotes the number of zeros of $p$ in $(a, b)$, counted with multiplicity, then

$$Z(p \mid (a, b)) \leqslant S^-(p(a), p'(a),..., p^{(r)}(a)) - S^+(p(b), p'(b),..., p^{(r)}(b)). \tag{3.2}$$

We now prove two lemmas for functions $f$ in $\zeta$.

LEMMA 1. *Suppose $t_{i-1} < t_i = t_{i+\mu-1} < t_{i+\mu}$. Suppose further that $f$ in $\zeta$ has exact degrees $p \geqslant 0$ and $q \geqslant 0$ on $(t_{i-1}, t_i)$ and $(t_{i+\mu-1}, t_{i+\mu})$, respectively, and has a point zero of multiplicity $m \geqslant 0$ at $t_i$. Then*

$$S^-(f(t_i^+),..., f^{(q)}(t_i^+)) - S^+(f(t_i^-),..., f^{(p)}(t_i^-)) \leqslant \mu - m. \tag{3.3}$$

*Proof.* For simplicity we shall write for $0 \leqslant j \leqslant n$,

$$S_j^- := S^-(f(t_i^+),..., f^{(j)}(t_i^+)),$$
$$S_j^+ := S^+(f(t_i^-),..., f^{(j)}(t_i^-)).$$

Suppose $l$ and $r$ are the least nonnegative integers with $f^{(l)}(t_i^-) \neq 0$ and $f^{(r)}(t_i^+) \neq 0$, respectively. Then $S_q^- \leqslant q-r$ and $S_p^+ \geqslant l$.

If $\min(r, l) \geqslant n+1-\mu$, then

$$S_q^- - S_p^+ \leqslant q-r-l \leqslant n-\min(r, l) - \max(r, l) \leqslant \mu - 1 - \max(r, l) \leqslant \mu - m$$

which gives (3.3).

Now suppose $\min(r, l) < n+1-\mu$. Then from (3.1) we see that $l=r=m$. We shall prove by induction that for $j=m,..., \min(n-\mu, p, q)$,

$$S_j^- - S_j^+ \leqslant \begin{cases} -m & \text{if } f^{(j)}(t_i^+) f^{(j)}(t_i^-) > 0, \\ -m-1 & \text{otherwise.} \end{cases} \tag{3.4}$$

Note that $S_m^- = 0$ and $S_m^+ = m$. Since from (3.1), $f^{(m)}(t_i^+) = \beta_m(t_i) f^{(m)}(t_i^-)$, we see that (3.4) is satisfied for $j=m$. Now assume that (3.4) is true for some $j$, $m \leqslant j < \min(n-\mu, p, q)$.

*Case* 1. $f^{(j)}(t_i^+)f^{(j)}(t_i^-)>0$.

If $f^{(j)}(t_i^-)f^{(j+1)}(t_i^-)>0$, then by (3.1), $f^{(j+1)}(t_i^+)f^{(j+1)}(t_i^-)>0$. Thus $f^{(j+1)}(t_i^+)f^{(j)}(t_i^+)>0$ and so

$$S_{j+1}^- - S_{j+1}^+ = S_j^- - S_j^+ \leqslant -m.$$

Hence (3.4) is true for $j+1$. Next suppose $f^{(j)}(t_i^-)f^{(j+1)}(t_i^-)\leqslant 0$. Then $S_{j+1}^+ = S_j^+ + 1$. If $f^{(j+1)}(t_i^+)f^{(j)}(t_i^+)\geqslant 0$, then $S_{j+1}^- = S_j^-$ and so

$$S_{j+1}^- - S_{j+1}^+ = S_j^- - S_j^+ - 1 \leqslant -m-1,$$

which gives (3.4) for $j+1$. If $f^{(j+1)}(t_i^+)f^{(j)}(t_i^+)<0$, then

$$S_{j+1}^- - S_{j+1}^+ = S_j^- - S_j^+ \leqslant -m.$$

But by (3.1) we must have $f^{(j+1)}(t_i^+)f^{(j+1)}(t_i^-)>0$, and so again we have (3.4) for $j+1$.

*Case* 2. $f^{(j)}(t_i^+)f^{(j)}(t_i^-)\leqslant 0$.

In this case $S_j^- - S_j^+ \leqslant -m-1$. If $S_{j+1}^+ = S_j^+ + 1$, then

$$S_{j+1}^- - S_{j+1}^+ \leqslant S_j^- + 1 - S_j^+ - 1 \leqslant -m-1,$$

and so (3.4) holds for $j+1$. Otherwise we must have $f^{(j+1)}(t_i^-)\neq 0$ and $f^{(j+1)}(t_i^-)f^{(j)}(t_i^-)\geqslant 0$. Then from (3.1) we must have $f^{(j+1)}(t_i^+)$ $f^{(j+1)}(t_i^-)>0$. Since

$$S_{j+1}^- - S_{j+1}^+ \leqslant S_j^- + 1 - S_j^+ \leqslant -m,$$

we see again that (3.4) holds for $j+1$.

We have thus established (3.4) and we next claim that for $j=m,\dots,$ $\min(n-\mu, q)$,

$$S_j^- - S_p^+ \leqslant -m. \tag{3.5}$$

If $p\geqslant \min(n-\mu, q)$, this follows immediately from (3.4). So suppose that $p<\min(n-\mu, q)$. Then (3.4) shows that (3.5) is true for $j=m,\dots, p$. If $f^{(p)}(t_i^+)f^{(p)}(t_i^-)\leqslant 0$, then

$$S_{p+1}^- - S_p^+ \leqslant S_p^- + 1 - S_p^+ \leqslant -m$$

by (3.4). Now by (3.1), $f^{(p+1)}(t_i^+)f^{(p)}(t_i^-)\geqslant 0$. So if $f^{(p)}(t_i^+)f^{(p)}(t_i^-)>0$, then $f^{(p+1)}(t_i^+)f^{(p)}(t_i^+)\geqslant 0$. Thus $S_{p+1}^- = S_p^-$ and so

$$S_{p+1}^- - S_p^+ = S_p^- - S_p^+ \leqslant -m.$$

Thus (3.5) holds for $j=p+1$. Now consider $j=p+2,\ldots,\min(n-\mu,q)$. It follows from (3.1) and $f^{(j)}(t_i^-)=f^{(j-1)}(t_i^-)=0$ that $f^{(j)}(t_i^+)=0$. Thus $S_j^-=S_{p+1}^-$ and so (3.5) holds.

If $q\leqslant n-\mu$, then (3.5) holds for $j=q$ which gives (3.3). If $q>n-\mu$, then (3.5) gives $S_{n-\mu}^- - S_p^+ \leqslant -m$ and since $S_q^- \leqslant S_{n-\mu}^- + \mu$, we again have (3.3).

LEMMA 2. *Suppose $t_i<t_j$ and for $f$ in $\zeta$, $f(t_i)^-=0=f(t_i)^+$ and $f$ does not have any interval zero in $(t_i,t_j)$. Then*

$$Z(f\mid(t_i,t_j))\leqslant|\{l: 0\leqslant l\leqslant n+k,\ t_i\leqslant t_l\leqslant t_j\}|-n-2. \tag{3.6}$$

*Proof.* Let $s_1<s_2<\cdots<s_r$ denote the distinct elements of $\{t_i,t_{i+1},\ldots,t_j\}$. Applying the Budan–Fourier theorem (3.2) to $f$ on $(s_l,s_{l+1})$, $1\leqslant l\leqslant r-1$, then gives

$$\begin{aligned}Z(f\mid(s_l,s_{l+1}))\leqslant{}& S^-(f(s_l^+),\ldots,f^{(p_l)}(s_l^+))\\ &-S^+(f(s_{l+1}^-),\ldots,f^{(p_l)}(s_{l+1}^-)),\end{aligned} \tag{3.7}$$

where $p_l$ is the exact degree of $f\mid(s_l,s_{l+1})$. Adding inequalities (3.7) for $l=1,\ldots,r-1$ and applying Lemma 1 gives

$$\begin{aligned}Z(f\mid t_i,t_j))\leqslant{}& S^-(f(t_i^+),\ldots,f^{(p_1)}(t_i^+))-S^+(f(t_j^-),\ldots,f^{(p_r)}(t_j^-))\\ &+|\{l: t_i<t_l<t_j\}|.\end{aligned} \tag{3.8}$$

Since $f(t_i)^-=0$, we see that $f^{(l)}(t_i^+)=0$ for $l=0,\ldots,n-\mu$, where $\mu$ is the multiplicity of $t_i$. Thus $S^-(f(t_i^+),\ldots,f^{(p_1)}(t_i^+))\leqslant\mu-1$. Similarly $f^{(l)}(t_j^-)=0$ for $l=0,\ldots,n-v$, where $v$ is the multiplicity of $t_j$. Hence $S^+(f(t_j^-),\ldots,f^{(p_r)}(t_j^-))\geqslant n-v+1$. Substituting these estimates into (3.8) gives (3.6). ∎

*Proof of Theorem* 1. We simply apply Lemma 2 between each pair of consecutive intervals on which $f$ vanishes identically and add the resulting inequalities.

COROLLARY 1. *If $k=0$, then the only element of $\zeta$ is the zero function.*

We can now deduce

THEOREM 2. *The dimension of $\zeta$ is $k$.*

*Proof.* For $i=0,\ldots,n+k$, let $\mu_i=|\{j\geqslant i: t_j=t_i\}|$ and define a function $\phi_i$ as follows.

(a) For $x<t_i$, $\phi_i(x)=0$.

(b) On each nontrivial interval $[t_j,t_{j+1})$ $(j=i,\ldots,n+k-1)$ and on $[t_{n+k},\infty)$, $\phi_i$ coincides with a polynomial of degree $\leqslant n$.

(c)

$$\phi_i^{(j)}(t_i^+) = 1, \text{ if } j = n - \mu_i + 1,$$
$$= 0, \text{ otherwise.}$$

(d) If $t$ has multiplicity $\mu$ in $\mathbf{t}$ and $t_i < t < t_{n+k}$, then $\phi_i$ satisfies (3.1) and

$$\phi_i^{(j)}(t^+) = \phi_i^{(j)}(t^-) \qquad (j = n - \mu + 1, ..., n).$$

(e) If $t_i < t_{n+k}$, then $\phi_i$ is $C^n$ in a neighbourhood of $t_{n+k}$.

Clearly $\phi_i$ is defined uniquely. Now a function $f$ lies in $\zeta$ if and only if it can be written in the form

$$f = \sum_{i=0}^{n+k} a_i \phi_i, \tag{3.9}$$

where

$$f^{(j)}(t_{n+k}^+) = 0 \qquad (j = 0, ..., n). \tag{3.10}$$

Thus to prove Theorem 2 it is sufficient to show that the matrix $\|\phi_i^{(j)}(t_{n+k}^+)\|_{i=0}^{n+k}\,{}_{j=0}^{n}$ has rank $n+1$. To do this it is sufficient to show that

$$\det \|\phi_i^{(j)}(t_{n+k}^+)\|_{i=k}^{n+k}\,{}_{j=0}^{n} \neq 0. \tag{3.11}$$

Now take any numbers $b_k, ..., b_{n+k}$ for which the function $f = \sum_k^{n+k} b_i \phi_i$ satisfies (3.10). Then to show (3.11) is equivalent to showing that $b_k, ..., b_{n+k}$ must all vanish. Now the function $f$ lies in $\zeta((t_k, ..., t_{n+k}))$ and so by Corollary 1, $f$ must vanish identically. Thus $0 = f^{(n-\mu_k+1)}(t_k^+) = b_k$. Continuing in this manner, we can also show that $b_{k+1} = \cdots = b_{n+k} = 0$, which completes the proof. ∎

To finish this section we consider interpolation by elements of $\zeta$. Take a nondecreasing sequence $\mathbf{x} = (x_i)_0^{k-1}$ with multiplicities at most $n+1$. We shall say the *interpolation problem* $(\zeta, \mathbf{x})$ *is solvable* if for every sequence $\mathbf{y} = (y_i)_0^{k-1}$ there is a unique function $f$ in $\zeta$ which interpolates $\mathbf{y}$ at $\mathbf{x}$; i.e.,

$$f^{(v_i)}(x_i) = y_i \qquad (i = 0, ..., k-1),$$

where

$$v_i = |\{j < i: x_j = x_i\}|. \tag{3.12}$$

For (3.12) to make sense we assume $f^{(v_i)}$ is continuous at $x_i$ for all $f$ in $\zeta$ $(i = 0, ..., k-1)$. The following result generalises a well-known result of Schoenberg and Whitney [7] for ordinary polynomial splines.

THEOREM 3. *The interpolation problem* $(\zeta, \mathbf{x})$ *is solvable if and only if*

$$t_i < x_i < t_{i+n+1} \qquad (i=0,..., k-1). \tag{3.13}$$

*Proof.* Suppose (3.13) is not satisfied. Then for some $i$ $(0 \leqslant i \leqslant k-1)$ either $x_i \leqslant t_i$ or $t_{i+n+1} \leqslant x_i$. First suppose $x_i \leqslant t_i$. By Theorem 2, $\zeta((t_i,..., t_{n+k}))$ has dimension $k-i$ and so there is a nontrivial element $f$ of $\zeta((t_i,..., t_{n+k}))$ which interpolates zero data at $x_{i+1},..., x_{k-1}$. Since $f$ vanishes on $(-\infty, t_i)$, it also interpolates zero data at $x_o,..., x_i$. Thus $(\zeta, \mathbf{x})$ is not solvable. If $t_{i+n+1} \leqslant x_i$, then we can similarly find a nontrivial element of $\zeta((t_0,..., t_{i+n+1}))$ which interpolates zero data at $\mathbf{x}$.

Now suppose (3.13) is satisfied. We suppose $f$ is a nontrivial element of $\zeta$ which interpolates zero data at $\mathbf{x}$ and reach a contradiction. Choose $a<b$ such that $f(a)^- = 0 = f(b)^+$ and $f$ does not vanish on any nontrivial interval in $(a, b)$. Define a function $g$ by

$$g(x) = \begin{cases} f(x), & a \leqslant x < b, \\ 0, & \text{elsewhere.} \end{cases}$$

We note that $a$ and $b$ must lie in $\mathbf{t}$ and we let $i = \min\{l: t_l = a\}$, $j = \max\{l: t_l = b\}$. Then $g$ lies in $\zeta((t_i,..., t_j))$ and by Theorem 1, $j-i \geqslant n+1$ and $Z(g) \leqslant j-i-n-1$. But by (3.13), $t_i < x_i \leqslant x_{j-n-1} < t_j$, and since $g$ interpolates zero data at $x_i,..., x_{j-n-1}$, we must have $Z(g) \geqslant j-n-i$. This gives the required contradiction. ∎

## 4. GENERALISED $\beta$-SPLINES

We shall now construct a basis for $\zeta$ of splines which have, in a sense, minimal support. Take an integer $i$, $0 \leqslant i \leqslant k-1$, and consider the space $\zeta_i := \zeta((t_i,..., t_{i+n+1}))$. By Theorem 1 any nontrivial element of $\zeta_i$ has no zeros and by Theorem 2 the space $\zeta_i$ has dimension 1. We let $M_i$ denote some nontrivial element of $\zeta_i$ satisfying $M_i(x) \geqslant 0$ for all $x$. Then $M_i$ is unique upto a positive normalising constant and we call it a *$\beta$-spline* with knots $t_i,..., t_{i+n+1}$ and parameters $B(t)$, $\Gamma(t)$ $(t \in \mathbf{t}, t_i < t < t_{i+n+1})$.

LEMMA 3. *A $\beta$-spline $M_i$ satisfies*

$$M_i(x) \begin{cases} >0 & \text{if} \quad t_i < x < t_{i+n+1}, \\ =0 & \text{if} \quad x < t_i \text{ or } x > t_{i+n+1}. \end{cases} \tag{4.1}$$

*Moreover if $t_i$ and $t_{i+n+1}$ have multiplicities $\mu$ and $\nu$, respectively, in $(t_i,..., t_{i+n+1})$, then*

$$M_i^{(n-\mu+1)}(t_i^+) > 0, \qquad (-1)^{n-\nu+1} M_i^{(n-\nu+1)}(t_{i+n+1}^-) > 0. \tag{4.2}$$

*Proof.* If either $M_i^{(n-\mu+1)}(t_i^+)$ or $M_i^{(n-\nu+1)}(t_{i+n+1}^-)$ were to vanish, then $M_i$ would lie in $\zeta(\mathbf{s})$ for a strict subsequence $\mathbf{s}$ of $\mathbf{t}$. Then Corollary 1 would imply that $M_i$ vanished identically, which is not the case. Since $M_i^{(j)}(t_i^+)=0$ for $j=0,..., n-\mu$, and $M_i(x)\geqslant 0$ for all $x$, we must have $M_i^{(n-\mu+1)}(t_i^+)>0$. Similarly we have $(-1)^{n-\nu+1} M_i^{(n-\nu+1)}(t_{i+n+1}^-)>0$.

Also by Theorem 1, $Z(M_i)=0$ and so we must have (4.1). ∎

THEOREM 4. *The $\beta$-splines $M_0,..., M_{k-1}$ form a basis for $\zeta$.*

*Proof.* By Theorem 2 it is sufficient to show that $M_0,..., M_{k-1}$ are linearly independent. Suppose that for constants $a_0,..., a_{k-1}$, the function $f:= a_0M_0+ \cdots +a_{k-1}M_{k-1}$ vanishes identically.

If $t_0$ has multiplicity $\mu$ in $\mathbf{t}$, then

$$0=f^{(n-\mu+1)}(t_0^+)=a_0M_0^{(n-\mu+1)}(t_0^+).$$

So by (4.2), $a_0=0$. Continuing in this manner gives $a_1= \cdots =a_{k-1}=0$ which completes the proof. ∎

We shall now show that $\beta$-splines share the well-known "variation-diminishing" property of $B$-splines.

THEOREM 5. *For any numbers $a_0,..., a_{k-1}$,* we have

$$V(a_0M_0+ \cdots +a_{k-1}M_{k-1})\leqslant S^-(a_0,..., a_{k-1}). \tag{4.3}$$

The proof of Theorem 5 follows the method of Lane and Riesenfield [5], which has been further developed in [4]. The crucial step is

LEMMA 4. *Suppose $\mathbf{t}=(t_i)_0^{n+l}$ is a nondecreasing sequence and $\hat{\mathbf{t}}=(\hat{t}_i)^{n+l+1}$ is a nondecreasing sequence with multiplicities at most $n+1$ which is gained by adding an extra element $s$ to $\mathbf{t}$, $t_0<s<t_{n+l}$. Suppose $\mathbf{t}$ has associated parameters $B$, $\Gamma$ and we define parameters $\hat{B}$, $\hat{\Gamma}$ for $\hat{\mathbf{t}}$ as follows.*

*If $t\neq s$, we put $\hat{B}(t)=B(t)$, $\hat{\Gamma}(t)=\Gamma(t)$. If $s$ is not in $\mathbf{t}$, we put $\hat{B}(s)=(1,..., 1)$, $\hat{\Gamma}(s)=(0,..., 0)$. If $s$ has multiplicity $\mu$ in $\mathbf{t}$, $1\leqslant\mu\leqslant n$, then we put $\hat{B}(s)=(B_j(s))_0^{n-\mu-1}$, $\hat{\Gamma}(s)=(\Gamma_j(s))_1^{n-\mu-1}$.*

*Now suppose $M_0,..., M_{l-1}$ are $\beta$-splines for $\zeta(\mathbf{t})$ and $\hat{M}_0,..., \hat{M}_l$ are $\beta$-splines for $\zeta(\hat{\mathbf{t}})$. If for constants $a_i$, $\hat{a}_i$,*

$$a_0M_0+ \cdots +a_{l-1}M_{l-1}=\hat{a}_0\hat{M}_0+ \cdots +\hat{a}_l\hat{M}, \tag{4.4}$$

*then*

$$S^-(\hat{a}_0,..., \hat{a}_l)\leqslant S^-(a_0,..., a_{l-1}). \tag{4.5}$$

*Proof.* Consider a $\beta$-spline $M_i$, $0 \leqslant i \leqslant l-1$. If $t_{i+n+1} \leqslant s$, then $M_i = \alpha_i \hat{M}_i$ for some $\alpha_i > 0$. If $s \leqslant t_i$, then $M_i = \delta_i \hat{M}_{i+1}$ for some $\delta_i > 0$. If $t_i < s < t_{i+n+1}$, then $M_i$ is in $\zeta((\hat{t}_i, ..., \hat{t}_{i+n+2}))$ and by Theorem 4,

$$M_i = \alpha_i \hat{M}_i + \delta_i \hat{M}_{i+1},$$

where by (4.2), $\alpha_i, \delta_i > 0$. So from (4.4) and Theorem 4 we get

$$\begin{aligned} &\hat{a}_0 = a_0 \alpha_0, && \hat{a}_l = a_{l-1} \delta_{l-1}, \\ &\hat{a}_i = a_{i-1} \delta_{i-1} + a_i \alpha_i && (i = 1, ..., l-1), \\ &\quad \text{where } \alpha_i \delta_i \geqslant 0 && (i = 0, ..., l-1). \end{aligned} \tag{4.6}$$

We shall deduce (4.5) from (4.6). Indeed we shall prove by induction on $j$ that for $j = 0, ..., l-1$,

$$S^-(\hat{a}_0, ..., \hat{a}_j, a_j) \leqslant S^-(a_0, ..., a_j). \tag{4.7}$$

Clearly (4.7) is true for $j = 0$. Assume it is true up to some $j$, $0 \leqslant j < l-1$.

Case 1: $a_j = 0$.

Let $m = \max\{i < j : a_i \neq 0\}$. Since $\hat{a}_{j+1} = a_{j+1} \alpha_{j+1}$, $\hat{a}_j = a_{j-1} \delta_{j-1}$, we have

$$S^-(\hat{a}_0, ..., \hat{a}_{j+1}, a_{j+1}) \leqslant S^-(\hat{a}_0, ..., \hat{a}_{j-1}, a_{j-1}, a_{j+1}).$$

Continuing in this way we get

$$\begin{aligned} S^-(\hat{a}_0, ..., \hat{a}_{j+1}, a_{j+1}) &\leqslant S^-(\hat{a}_0, ..., \hat{a}_m, a_m, a_{j+1}) \\ &= S^-(\hat{a}_0, ..., \hat{a}_m, a_m) + S^-(a_m, a_{j+1}) \\ &\leqslant S^-(a_0, ..., a_m) + S^-(a_m, a_{j+1}) \\ &= S^-(a_0, ..., a_{j+1}). \end{aligned}$$

*Case* 2: $a_j \neq 0$.

First suppose $\hat{a}_{j+1} a_j \geqslant 0$. Then

$$\begin{aligned} S^-(\hat{a}_0, ..., \hat{a}_{j+1}, a_{j+1}) &\leqslant S^-(\hat{a}_0, ..., \hat{a}_j, a_j, a_{j+1}) \\ &= S^-(\hat{a}_0, ..., \hat{a}_j, a_j) + S^-(a_j, a_{j+1}) \\ &\leqslant S^-(a_0, ..., a_j) + S^-(a_j, a_{j+1}) \\ &= S^-(a_0, ..., a_{j+1}). \end{aligned}$$

Next suppose $\hat{a}_{j+1}a_j<0$. Then $\hat{a}_{j+1}a_{j+1}>0$, $a_ja_{j+1}<0$ and so

$$\begin{aligned}S^-(\hat{a}_0,..., \hat{a}_{j+1}, a_{j+1}) &= S^-(\hat{a}_0,..., \hat{a}_j, a_{j+1})\\ &\leqslant S^-(\hat{a}_0,..., \hat{a}_j)+1\\ &\leqslant S^-(a_0,..., a_j)+1\\ &= S^-(a_0,..., a_{j+1}).\end{aligned}$$

Thus in all cases (4.7) holds for $j+1$. Hence (4.7) is true for $j=0,..., l-1$, and in particular for $j=l-1$ which gives (4.5). ∎

*Proof of Theorem* 5. Let $f=a_0M_0+\cdots a_{k-1}M_{k-1}$. Since $f$ vanishes outside $[t_0, t_{n+k})$ and is continuous on the right at $t_0$, $V(f)$ is the supremum of $S^-(f(x_1),..., f(x_r))$ over all sequences $(x_1,..., x_r)$ with $t_0<x_1<\cdots<x_r<t_{n+k}$, for all $r$. Let $(x_1,..., x_r)$ be such a sequence and let $\hat{\mathbf{t}}$ be the nondecreasing sequence gained by including $x_1,..., x_r$ in $\mathbf{t}$, each with multiplicity $n+1$. Associated with $\hat{\mathbf{t}}$ we define parameters $\hat{B}$, $\hat{\Gamma}$ by $\hat{B}(t)=B(t)$, $\hat{\Gamma}(t)=\Gamma(t)$ for $t\in\hat{\mathbf{t}}$ $t\notin\{x_1,..., x_r\}$. (If $t\in\{x_1,..., x_r\}$, then $t$ has multiplicity $n+1$ in $\hat{\mathbf{t}}$ and so no parameters are required.) Let $\hat{M}_0,..., \hat{M}_l$ denote a $\beta$-spline basis for $\zeta(\hat{\mathbf{t}})$. Since $f$ is in $\zeta(\hat{\mathbf{t}})$, there are constants $\hat{a}_0,..., \hat{a}_l$ with

$$f=\hat{a}_0\hat{M}_0+\cdots+\hat{a}_l\hat{M}_l.$$

Applying Lemma 4 successively gives

$$S^-(\hat{a}_0,..., \hat{a}_l)\leqslant S^-(a_0,..., a_{k-1}). \tag{4.8}$$

Now take any $i$, $1\leqslant i\leqslant r$, and choose $j$ with $x_i=\hat{t}_j=\hat{t}_{j+n}$. Then

$$f(x_i)=f(x_i^+)=\hat{a}_j\hat{M}_j(x_i^+).$$

Recalling (4.2) we see that if $f(x_i)\neq 0$, then $\hat{a}_jf(x_i)>0$. Thus

$$S^-(f(x_1),..., f(x_r))\leqslant S^-(\hat{a}_0,..., \hat{a}_l).$$

Hence from (4.8),

$$V(f)\leqslant S^-(a_0,..., a_{k-1}).\quad ∎$$

For application to computer-aided design a desirable property of $\beta$-splines is that they can be normalised to form a partition of unity (see Sect. 2). This can only be possible if $\zeta$ contains functions which are constant on $(t_0, t_{n+k})$. For this we require that

$$\beta_0(t)=1, \qquad \gamma_1(t)=O(t\in\mathbf{t},\ t_0<t<t_{n+k}). \tag{4.9}$$

THEOREM 6. *Suppose* (4.9) *is satisfied and* $t_0$ *and* $t_{n+k}$ *both have multiplicity* $n+1$ *in* $\mathbf{t}$. *Then there is a unique basis for* $\zeta$ *of functions* $N_0,\ldots, N_{k-1}$ *satisfying the following conditions.*

*For* $i=0,\ldots, k-1$,

$$N_i(x)\begin{cases} >0 & \text{if} \quad t_i < x < t_{i+n+1}, \\ =0 & \text{if} \quad x < t_i \text{ or } x > t_{i+n+1}, \end{cases} \tag{4.10}$$

$$\sum_{i=0}^{k-1} N_i(x) = 1 \qquad (t_0 \leqslant x < t_{n+k}). \tag{4.11}$$

*Furthermore for* $i=0,\ldots, k-1$, *the function* $N_i$ *depends only on* $t_i,\ldots, t_{i+n+1}$ *and on* $B(t)$, $\Gamma(t)$ *for* $t_i < t < t_{i+n+1}$.

*Proof.* If $n=0$, we define

$$N_i(x) = \begin{cases} 1, & t_i \leqslant x < t_{i+1}, \\ 0, & \text{elsewhere.} \end{cases}$$

Clearly this satisfies the conditions of the theorem and so we may henceforth assume $n \geqslant 1$. From our earlier work we see that (4.10) is satisfied if and only if $N_i = a_i M_i$ for some $a_i > 0$. Now $\zeta$ clearly contains the function

$$F(x) = \begin{cases} 1, & t_0 \leqslant x < t_{n+k}, \\ 0, & \text{elsewhere.} \end{cases}$$

By Theorem 4 there are unique constants $a_0,\ldots, a_{k-1}$ with $\sum_0^{k-1} a_i M_i = F$. Thus to show (4.10), (4.11) is equivalent to showing $a_i > 0$ for $i=0,\ldots, k-1$.

We now fix $i$, $0 \leqslant i \leqslant k-1$ and let $f = \sum_0^i a_j M_j$. Then

$$f(x) = 1 \qquad \text{for } t_0 \leqslant x < t_{i+1}, \tag{4.12}$$

$$f(x) = 0 \qquad \text{for } x \geqslant t_{i+n+1}. \tag{4.13}$$

First suppose that $t_{i+1} = t_{i+n+1}$. Then $1 = f(t_{i+1}^-) = a_i M_i(t_{i+n+1}^-)$. By (4.2), $M_i(t_{i+n+1}^-) > 0$ and so $a_i > 0$.

Next suppose $t_{i+1} < t_{i+n+1}$. In this case,

$$f(t_{i+n+1}^-) = f(t_{i+n+1}^+) = 0. \tag{4.14}$$

Let $\nu$ be the largest integer with $t_{i+\nu} = t_i$. Since $f = F - \sum_{i+1}^{k-1} a_j M_j$, we see that

$$\begin{aligned} f(t_i^+) &= 1, \\ f^{(j)}(t_i^+) &= 0 \qquad \text{for} \quad j=1,\ldots, n-\nu. \end{aligned} \tag{4.15}$$

If $\nu > 0$, then

$$f(t_{i+1}^+) = f(t_i^+) = 1. \tag{4.16}$$

If $\nu = 0$, we have $t_i < t_{i+1} < t_{i+n+1}$ and so, recalling (4.12), we see that

$$f(t_{i+1}^+) = f(t_{i+1}^-) = 1. \tag{4.17}$$

Now by (4.13) and (4.15), $f'$ is in the space $\zeta((t_{i+1},..., t_{i+n+1}), \hat{B}, \hat{\Gamma})$, where for any knot $t$ with multiplicity $\mu$ and $t_{i+1} < t < t_{i+n+1}$,

$$\hat{B}_j(t) = B_{j+1}(t) \qquad (j = 0,..., n-1-\mu),$$

$$\hat{\Gamma}_j(t) = \Gamma_{j+1}(t) \qquad (j = 1,..., n-1-\mu).$$

So from Theorem 1 we see that $Z(f') = 0$. Since from (4.16), (4.17), and (4.13) we have $f(t_{i+1}^+) = 1$, $f(t_{i+n+1}^-) = 0$, we must have

$$f'(x) < 0 \qquad \text{for} \quad t_{i+1} < x < t_{i+n+1}.$$

Furthermore (4.2) tells us that if $\eta$ is the multiplicity of $t_{i+n+1}$ in $(t_{i+1},..., t_{i+n+1})$, then

$$(-1)^{n-\eta} (f')^{(n-\eta)} (t_{i+n+1}^-) < 0.$$

Thus we have

$$0 < (-1)^{n-\eta+1} f^{(n-\eta+1)}(t_{i+n+1}^-) = (-1)^{n-\eta+1} a_i M_i^{(n-\eta+1)}(t_{i+n+1}^-),$$

and applying (4.2) shows that $a_i > 0$.

It remains to show that for given $i$, $0 \leqslant i \leqslant k-1$, the function $N_i$ depends only on $t_i,..., t_{i+n+1}$ and on $B(t)$, $\Gamma(t)$ for $t_i < t < t_{i+n+1}$. Suppose $\hat{\mathbf{t}} = (\hat{t}_j)_0^{n+k}$ is another nondecreasing sequence with multiplicities at most $n+1$, and $\hat{t}_j = t_j$ for $j = 0,..., i+n+1$. Suppose further that for any knot $t$ in $\hat{\mathbf{t}}$ with $\hat{t}_0 < t < \hat{t}_{n+k}$ we have associated parameters $\hat{B}(t)$, $\hat{\Gamma}(t)$, where $\hat{B}(t) = B(t)$, $\hat{\Gamma}(t) = \Gamma(t)$ for $t < t_{i+n+1}$. We assume $\hat{t}_{n+k}$ has multiplicity $n+1$ and for each $t$, $\hat{B}_0(t) = 1$, $\hat{\Gamma}_1(t) = 0$. Then as above we can construct the normalised $\beta$-splines $\hat{N}_0,..., \hat{N}_{k-1}$ for the space $\zeta(\hat{\mathbf{t}}\, \hat{B}, \hat{\Gamma})$.

From our earlier work we see that for $j = 0,..., i$, $\hat{N}_j = c_j N_j$ for some constant $c_j > 0$. Since both $(N_j)_0^{k-1}$ and $(\hat{N}_j)_0^{k-1}$ form partitions of unity, we have for $t_0 \leqslant x < t_{i+n+1}$,

$$\begin{aligned} 0 &= \sum_{j=0}^{k-1} N_j(x) - \sum_{j=0}^{k-1} \hat{N}_j(x) \\ &= \sum_{j=0}^{i} (1-c_j)\, N_j(x) - \sum_{j=i+1}^{k-1} (N_j(x) - \hat{N}_j(x)). \end{aligned}$$

Proceeding exactly as in the proof of Theorem 4 we see that $1-c_j=0$ for $j=0,\ldots,i$. In particular $\hat{N}_i=N_i$ and so $N_i$ is independent of $t_j$ for $j>i+n+1$ and of $B(t)$, $\Gamma(t)$ for $t\geqslant t_{i+n+1}$. By a similar argument, we can show $N_i$ is independent of $t_j$ for $j<i$ and of $B(t)$, $\Gamma(t)$ for $t\leqslant t_i$. ∎

Thus to any numbers $t_0\leqslant t_1\leqslant\cdots\leqslant t_{n+1}$ with $t_0<t_{n+1}$ and parameters $B(t)$, $\Gamma(t)$ for $t$ in $\{t_0,\ldots,t_{n+1}\}$ with $t_0<t<t_{n+1}$, satisfying (4.9) we can associate a unique normalised $\beta$-spline $N$. Now take $\mathbf{t}, B, \Gamma$ as in Theorem 6 except that we drop the condition that $t_0$ and $t_{n+k}$ have multiplicity $n+1$. For $i=0,\ldots,k-1$ we denote by $N_i$ the normalised $\beta$-spline with knots $t_i,\ldots,t_{i+n+1}$. By adding extra knots at $t_0$ and $t_{n+1}$, we can create a partition of unity as in Theorem 6. In particular, we see that if $t_n<t_k$,

$$\sum_{i=0}^{k-1} N_i(x)=1 \qquad \text{for} \quad t_n\leqslant x<t_k. \tag{4.18}$$

*Remark.* Let $N$ be the normalised $\beta$-spline with knots $t_0,\ldots,t_{n+1}$ and parameters $B, \Gamma$. Suppose that $t_0<t_1$. Then extending to a partition of unity as in Theorem 6 we see that for $t_0\leqslant x<t_1$, $N(x)=1-f(x)$, where $f$ is a sum of $\beta$-splines with knots upto $t_n$. Thus for $t_0\leqslant x<t_1$, $N(x)$ is independent of $t_{n+1}$, $B(t_n)$, and $\Gamma(t_n)$. Similarly if $t_n<t_{n+1}$, then for $t_n\leqslant x<t_{n+1}$, $N(x)$ is independent of $t_0$, $B(t_1)$, and $\Gamma(t_1)$.

## 5. Geometric Continuity

The interest in the space $\zeta$ of splines considered in Sections 3 and 4 lies, as described in Section 2, in the construction of a design curve $Q(t)=(Q_1(t), Q_2(t))$, where $Q_1, Q_2\in\zeta$. It is natural, therefore, to examine the geometric smoothness of such a curve. First we must define what is meant by geometric continuity of degree $n$.

Suppose $Q$ is a continuous function from an interval $[a-\delta, a+\delta]$ to $\mathbb{R}^p$ for some numbers $a$, $\delta>0$ and integer $p\geqslant 2$. We assume that for some $n\geqslant 1$, $Q$ is $C^n$ on $[a-\delta, a]$ and on $[a, a+\delta]$, and $Q'(t)\neq 0$ on $[a-\delta, a]$ and $[a, a+\delta]$.

We denote by $s$ the arc length

$$s(t)=\int_{a-\delta}^{t}\left\{\sum_{1}^{p} Q_i'(x)^2\right\}^{1/2} dx=\int_{a-\delta}^{t} |Q'(x)|\, dx. \tag{5.1}$$

We can reparametrise $Q$ by arc length, and we denote this function by $\hat{Q}$, i.e.,

$$\hat{Q}(s(t))=Q(t) \qquad (a-\delta\leqslant t\leqslant a+\delta).$$

Since $s$ is the natural geometric parameter for the curve $Q$, it is natural to say that the curve $Q$ has *geometric continuity of degree n* if $\hat{Q}$ is $C^n$ on $[0, s(a+\delta)]$. Since $\hat{Q}$ is clearly $C^n$ on $[0, s(a)]$ and $[s(a), s(a+\delta)]$, this is equivalent to

$$\hat{Q}^{(j)}(s(a)^+) = \hat{Q}^{(j)}(s(a)^-) \qquad (j = 1,..., n). \tag{5.2}$$

We note that $\hat{Q}'(s(t)) = Q'(t)/|Q'(t)|$ is the unit tangent vector to the curve. Moreover, $\hat{Q}''(s(t))$ denotes the curvature vector and thus second degree geometric continuity is equivalent to continuity of the unit tangent and the curvature (see [1]).

Clearly if $Q$ is $C^n$, then $Q$ has geometric continuity of degree $n$. The converse is not true, however, as the following result shows.

THEOREM 7. *The curve $Q$ has geometric continuity of degree $n$ if and only if there are constants $\beta_1,..., \beta_n$, $\beta_1 > 0$, such that for $k = 1,..., n$,*

$$Q^{(k)}(a^+) = \sum_{j=1}^{k} A_{k,j} Q^{(j)}(a^-), \tag{5.3}$$

$$A_{k,j} = \sum \left[ k_1, \overset{k}{\ldots}, k_j \right] \beta_{k_1} \cdots \beta_{k_j}, \tag{5.4}$$

*where the sum in* (5.4) *is over all sets $\{k_1,..., k_j\}$ of positive integers with $k_1 + \cdots + k_j = k$.*

In (5.4), $[k_1, \overset{k}{\ldots}, k_j]$ denotes the number of distinct partitions of a set of $k$ distinct elements into $j$ subsets comprising $k_1,..., k_j$ elements, respectively. Explicitly, if $\{k_1,..., k_j\}$ comprises $r$ distinct elements with multiplicities $m_1,..., m_r$, then

$$\left[ k_1, \overset{k}{\ldots}, k_j \right] = \frac{k!}{k_1! \cdots k_j! m_1! \cdots m_r!}. \tag{5.5}$$

We shall need a lemma, but first we list explicitly the conditions for geometric continuity upto degree 4.

$$\begin{aligned}
Q'(a^+) &= \beta_1 Q'(a^-), \\
Q^{(2)}(a^+) &= \beta_1^2 Q^{(2)}(a^-) + \beta_2 Q'(a^-), \\
Q^{(3)}(a^+) &= \beta_1^3 Q^{(3)}(a^-) + 3\beta_1\beta_2 Q^{(2)}(a^-) + \beta_3 Q'(a^-), \\
Q^{(4)}(a^+) &= \beta_1^4 Q^{(4)}(a^-) + 6\beta_1^2\beta_2 Q^{(3)}(a^-) \\
&\quad + (3\beta_2^2 + 4\beta_1\beta_3)\, Q^{(2)}(a^-) + \beta_4 Q'(a^-).
\end{aligned}$$

LEMMA 5. *Suppose for some intervals $I$, $J$ in $\mathbb{R}$, the functions $f: I \to J$, $G: J \to \mathbb{R}^p$ are $C^k$, $k \geqslant 1$. Then with $F(t) = G(f(t))$ $(t \in I)$, we have for $t$ in $I$,*

$$F^{(k)}(t) = \sum_{j=1}^{k} B_{k,j}(t)\, G^{(j)}(f(t)), \tag{5.6}$$

$$B_{k,j}(t) = \sum \left[ k_1, \overset{k}{\ldots}, k_j \right] f^{(k_1)}(t) \cdots f^{(k_j)}(t), \tag{5.7}$$

*where the summation is as in* (5.4).

*Proof.* We first derive a recurrence relation for $B_{k,j}$. Let $\Pi_{k,j}$ denote the set of all partitions of a set $\{a_1, ..., a_k\}$ into $j$ subsets. For $P$ in $\Pi_{k,j}$ we let $P_1, ..., P_j$ denote the elements of the partition and $p_1, ..., p_j$ the number of elements in $P_1, ..., P_j$, respectively. Then

$$B_{k,j} = \sum_{P \in \Pi_{k,j}} f^{(p_1)} \cdots f^{(p_j)}. \tag{5.8}$$

Now $Q$ is in $\Pi_{k+1,j}$ if and only if $Q = \{P_1, ..., P_{j-1}, \{a_{k+1}\}\}$ for some $P$ in $\Pi_{k,j-1}$, or $Q = \{P_1, ..., P_i \cup \{a_{k+1}\}, P_{i+1}, ..., P_j\}$ for some $i$, $1 \leqslant i \leqslant j$. Thus we have from (5.8),

$$\begin{aligned} B_{k+1,j} &= \sum_{P \in \Pi_{k,j-1}} f^{(p_1)} \cdots f^{(p_j)} f' \\ &\quad + \sum_{p \in \Pi_{k,j}} \sum_{i=1}^{j} f^{(p_1)} \cdots f^{(p_i+1)} \cdots f^{(p_j)} \\ &= f' B_{k,j-1} + B'_{k,j}, \end{aligned} \tag{5.9}$$

where we adopt the convention

$$B_{k,j} = 0 \qquad \text{for} \quad j \leqslant 0 \text{ or } j \geqslant k+1. \tag{5.10}$$

We can now prove (5.6) by induction on $k$. Since $B_{1,1} = f'$, (5.6) is true for $k = 1$. Assuming (5.6) for $k$ and differentiating gives

$$\begin{aligned} F^{(k+1)}(t) &= \sum_{j=1}^{k} \{B'_{k,j}(t)\, G^{(j)}(f(t)) + B_{k,j}(t)\, G^{(j+1)}(f(t)) f'(t)\} \\ &= \sum_{j=1}^{k+1} \{B'_{k,j}(t) + B_{k,j-1}(t) f'(t)\}\, G^{(j)}(f(t)) && \text{by (5.10)} \\ &= \sum_{j=1}^{k+1} B_{k+1,j}(t)\, G^{(j)}(f(t)) && \text{by (5.9),} \end{aligned}$$

which gives (5.6) for $k+1$, so completing the induction proof. ∎

*Proof of Theorem* 7. First, we suppose $Q$ satisfies (5.3), (5.4) for $k=1,\ldots,n$ and prove that $Q$ has geometric continuity of degree $n$. Choose a $C^n$ function $f\colon [a-\delta, a] \to [a-\delta, a]$ satisfying

$$f(a)=a, \qquad f^{(j)}(a)=\beta_j \qquad (j=1,\ldots,n).$$

We now reparametrise $Q$ by

$$\tilde{Q}(t)=\begin{cases} Q(f(t)), & a-\delta \leqslant t \leqslant a, \\ Q(t), & a \leqslant t \leqslant a+\delta. \end{cases}$$

Then by (5.6), (5.7), and (5.4), we see that for $k=1,\ldots,n$,

$$\tilde{Q}^{(k)}(a^-)=\sum_{j=1}^{k} A_{k,j} Q^{(j)}(f(a))^-),$$

and so from (5.3).

$$\tilde{Q}^{(k)}(a^+)=Q^{(k)}(a^+)=\tilde{Q}^{(k)}(a^-).$$

Thus $\tilde{Q}$ is $C^n$. So $\tilde{Q}$ has geometric continuity of degree $n$, and hence so does $Q$.

Conversely, we now suppose $Q$ has geometric continuity of degree $n$ and show that $Q$ satisfies (5.3), (5.4) for $k=1,\ldots,n$. As before we let $\hat{Q}$ denote $Q$ reparametrised by arc length $s(t)$, and we know $\hat{Q}$ is $C^n$ on $[0, s(a+\delta)]$. Now $s$ is a $C^n$ function on $[a, a+\delta]$, and we extend it to a $C^n$ function $f\colon [a-\delta, a+\delta] \to [0, s(a+\delta)]$. We then reparametrise $\hat{Q}$ by

$$\tilde{Q}(t)=\hat{Q}(f(t)) \qquad (a-\delta \leqslant t \leqslant a+\delta).$$

Thus $\tilde{Q}$ is a $C^n$ function on $[a-\delta, a+\delta]$ and

$$\tilde{Q}(t)=\hat{Q}(f(t))=\hat{Q}(s(t))=Q(t) \qquad (a \leqslant t \leqslant a+\delta). \tag{5.11}$$

Since $s$ is a $C^n$ function on $[a-\delta, a]$ with $s'(t)=|Q'(t)|>0$, $s^{-1}$ is a $C^n$ function on $[0, s(a)]$. Writing $g(t)=s^{-1}(f(t))$ $(a-\delta \leqslant t \leqslant a)$ we see that $g$ is $C^n$ and

$$\tilde{Q}(t)=\hat{Q}(f(t))=Q(s^{-1}(f(t)))=Q(g(t)) \qquad (a-\delta \leqslant t \leqslant a). \tag{5.12}$$

Then from (5.11), (5.12), (5.6), and (5.7), we have for $k=1,\ldots,n$,

$$Q^{(k)}(a^+)=\tilde{Q}^{(k)}(a^+)=\tilde{Q}^{(k)}(a^-)=\sum_{j=1}^{k} C_{k,j} Q^{(j)}(a^-),$$

where $C_{k,j}=\sum [k_1,\overset{k}{\ldots}, k_j]\, g^{(k_1)}(a)\cdots g^{(k_j)}(a^-)$. Writing $\beta_j=g^{(j)}(a^-)$ $(j=1,\ldots,n)$ gives (5.3), (5.4). ▌

## 6. The Design Curve Revisited

The normalised $\beta$-splines of Section 4 can be used to construct a design curve in exactly the same manner as for the special case of cubic $\beta$-splines as described in Section 2. To be precise, we take control vertices $V_i=(x_i, y_i)$ $(i=0,..., k-1)$ and define

$$Q(t)=\sum_{i=0}^{k-1} V_i N_i(t) \qquad (t_n \leqslant t < t_k). \tag{6.1}$$

We consider again the properties of the design curve considered in Section 2.

(a) *Smoothness.* We would like the curve $Q$ to satisfy geometric continuity of degree $n-1$ at a simple knot $a$. It is easily seen from Theorem 7 that this is possible only if for some $\beta_1>0$, $B(a)=(1, \beta_1, \beta_1^2,..., \beta_1^{n-1})$. For $n \leqslant 2$ or $n \leqslant 5$ we must also have $\Gamma(a)=0$, in which case the $\beta$-splines reduce to $B$-splines by a simple change of scale on each subinterval.

For $n=3$ we have geometric continuity of degree 2 only if for some $\beta_2 \geqslant 0$, $\Gamma(a)=(0, \beta_2)$. In this case the $\beta$-splines are the same as those considered in Section 2 except that we can allow different values of $\beta_1$ and $\beta_2$ at different knots. For $n=4$, we have geometric continuity of degree 3 only if for some $\beta_2 \geqslant 0$, $\Gamma(a)=(0, \beta_2, 3\beta_1\beta_2)$.

The above remarks can be easily modified to cover the case of a multiple knot.

(b) *Approximation.* From (4.10) and (4.18) we see that for $t$ in $(t_j, t_{j+1})$, $n \leqslant j \leqslant k-1$,

$$Q(t)=\sum_{i=j-n}^{j} V_i N_i(t), \qquad \sum_{i=j-n}^{j} N_i(t)=1.$$

Thus $Q(t)$ lies in the convex hull of $V_{j-n},..., V_j$.

(c) *Shape preservation.* Because the $\beta$-splines are variation-diminishing (Theorem 5), we can show exactly as in Section 2 that the design curve crosses any given straight line no more often than does the control polygon.

(d) *Local control.* From (4.10), we see that changing a control vertex $V_i$ will affect $Q(t)$ only for $t$ in $(t_i, t_{i+n+1})$. We can also change a knot $t_i$, $0 \leqslant i \leqslant k+n$. This will affect the $\beta$-splines $\{N_j: i-n-1 \leqslant j \leqslant i, 0 \leqslant j \leqslant k-1\}$ and, recalling the remark at the end of Section 4, we see that this will affect $Q(t)$ only for $t$ in $[t_n, t_k]$ with $t_{i-n} < t < t_{i+n}$. (In particular $Q$ is independent of $t_0$ and $t_{n+k}$.)

The advantage of $\beta$-splines over $B$-splines is that we can also alter the shape of the curve by altering a parameter $B(t_i)$ or $\Gamma(t_i)$ $(1 \leqslant i \leqslant n+k-1)$. This will affect the $\beta$-splines $\{N_j: i-n \leqslant j \leqslant i-1, 0 \leqslant j \leqslant k-1\}$ and will affect $Q(t)$ only for $t$ in $[t_n, t_k]$ with $t_{i-n+1} < t < t_{i+n-1}$. (In particular $Q$ is independent of $B(t_1)$, $\Gamma(t_1)$, $B(t_{n+k-1})$ and $\Gamma(t_{n+k-1})$.)

To finish this paper we shall give an explicit formula for the normalised cubic $\beta$-spline $N_0$ with simple knots satisfying geometric continuity of degree 2. Thus for $i=1, 2, 3$, there are numbers $\beta_1(i) > 0$, $\beta_2(i) \geqslant 0$ such that $B(t_i) = (1, \beta_1(i), \beta_1(i)^2)$, $\Gamma(t_i) = (0, \beta_2(i))$. By making a suitable change of scale on each subinterval $(t_i, t_{i+1})$ we may assume without loss of generality that the knots are uniformly spaced with

$$t_{i+1} = t_i + 1 \qquad (i = 0, ..., 3).$$

In [2] the formula is given for the special case

$$\beta_1(1) = \beta_1(2) = \beta_1(3), \qquad \beta_2(1) = \beta_2(2) = \beta_2(3). \tag{6.2}$$

To derive this formula it will be useful to extend the knot sequence to $\mathbf{t} = (t_i)_{i=-3}^{5}$ (where the values of the extra knots and parameters are immaterial). For $j = -3, ..., 1$, we let $N_j$ denote the normalised $\beta$-spline with knots $t_j, ..., t_{j+4}$. Then we denotė by $f$ the function

$$f(x) = N_{-3}(x) + N_{-2}(x) + N_{-1}(x) \qquad (t_0 \leqslant x \leqslant t_3).$$

Clearly

$$f^{(j)}(t_3) = 0 \qquad (j = 0, 1, 2). \tag{6.3}$$

Now

$$f(x) = 1 - N_0(x) \qquad (t_0 \leqslant x \leqslant t_1) \tag{6.4}$$

and so

$$f(t_0) = 1, \qquad f'(t_0) = f''(t_0) = 0. \tag{6.5}$$

Then from (6.3), (6.5), and the conditions (3.1) at $t_1$ and $t_2$, we can find $f$ by solving three linear equations in three unknowns. By (6.4), this gives $N_0(x)$ for $t_0 \leqslant x \leqslant t_1$. Replacing $\beta_1(i)$, $\beta_2(i)$ by $\beta_1(i+1)$, $\beta_2(i+1)$ $(i = 1, 2)$, then gives $N_1(x)$ for $t_1 \leqslant x \leqslant t_2$, and we have

$$N_0(x) = 1 - f(x) - N_1(x) \qquad (t_1 \leqslant x \leqslant t_2).$$

Having derived $N_0(x)$ for $t_0 \leqslant x \leqslant t_2$, we can easilt find $N_0(x)$ for $t_2 \leqslant x \leqslant t_4$ by considering the reflection $x \to -x$. The resulting formula is more neatly expressed by replacing the parameter $\beta_2(i)$ by a parameter

$$\alpha_i = \beta_1(i)^2 + \beta_1(i) + \tfrac{1}{2}\beta_2(i) \qquad (i = 1, 2, 3).$$

Writing, for simplicity, $\beta_1(i) = \beta_i$ $(i = 1, 2, 3)$, we have

$$
\begin{aligned}
N_0(x) &= \delta_1^{-1}\alpha_2 u^3, \; u = x - t_0, \; t_0 \leqslant x \leqslant t_1, \\
&= \delta_1^{-1}\alpha_2 + \delta_1^{-1}3\alpha_2\beta_1 u + \delta_1^{-1}3\alpha_2(\alpha_1 - \beta_1)\, u^2 \\
&\quad + \delta_1^{-1}(\alpha_2\beta_1 + \alpha_1\beta_2^2 - 2\alpha_1\alpha_2)\, u^3 - \delta_2^{-1}\alpha_3 u^3, \\
&\qquad u = x - t_1, \; t_1 \leqslant x \leqslant t_2, \\
&= \delta_2^{-1}\alpha_2\beta_3^3 + \delta_2^{-1}3\alpha_2\beta_3^2 u + \delta_2^{-1}3\alpha_2(a_3 - \beta_3^2)\, u^2 \\
&\quad + \delta_2^{-1}(\alpha_3\beta_2 + \alpha_2\beta_3^2 - 2\alpha_2\alpha_3)\, u^3 - \delta_1^{-1}\alpha_1\beta_2^3 u^3, \\
&\qquad u = t_3 - x, \; t_2 \leqslant x \leqslant t_3, \\
&= \delta_2^{-1}\alpha_2\beta_3^3 u^3, \qquad u = t_4 - x, \; t_3 \leqslant x \leqslant t_4,
\end{aligned}
$$

where

$$\delta_i = \alpha_i(\beta_{i+1}^3 + \beta_{i+1}^2) + \alpha_{i+1}(\beta_i + 1) + \alpha_i\alpha_{i+1} \qquad (i = 1, 2).$$

It can be checked that if (6.2) holds then this formula reduces to that in [2, p. 10].

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